MTH 166

Lecture-23

Revision of Unit-4 and MCQ Practice

Methods of Formation of PDE:

1. By elimination of arbitrary constants

Problem. **Form the PDE for:** $\boldsymbol{u = ax + by,}$ $\quad$ ***a*** **and** ***b*** **are constants**

Solution. Given $u = ax + by$ (1) $[u = f(x, y)]$

Differentiating (1) partially w.r.t *x*

$$\frac{\partial u}{\partial x} = a(1) + b(0) = a \qquad \Rightarrow p = a$$

Differentiating (1) partially w.r.t *y*

$$\frac{\partial u}{\partial y} = a(0) + b(1) = b \qquad \Rightarrow q = b$$

Putting values of *a* and *b* in equation (1):

$u = px + qy$ **or** $u = \left(\frac{\partial u}{\partial x}\right)x + \left(\frac{\partial u}{\partial y}\right)y$ which is the required PDE.

Methods of Formation of PDE:

2. By elimination of arbitrary functions

Problem. **Form the PDE for:** $\boldsymbol{u = f(x^2 + y^2)}$

Solution. Given $u = f(x^2 + y^2)$ (1) $[u = f(x, y)]$

Differentiating (1) partially w.r.t x

$$\frac{\partial u}{\partial x} = f'(x^2 + y^2)(2x) \Rightarrow p = f'(x^2 + y^2)(2x) \Rightarrow \frac{p}{2x} = f'(x^2 + y^2) \qquad (2)$$

Differentiating (1) partially w.r.t y

$$\frac{\partial u}{\partial y} = f'(x^2 + y^2)(2x) \Rightarrow q = f'(x^2 + y^2)(2y) \Rightarrow \frac{q}{2y} = f'(x^2 + y^2) \qquad (3)$$

Comparing (2) and (3), we get: $\frac{p}{2x} = \frac{q}{2y}$

$\Rightarrow py = qx$ **or** $\left(\frac{\partial u}{\partial x}\right) y = \left(\frac{\partial u}{\partial y}\right) x$ which is the required PDE.

Classification/Nature of Partial Differential Equations:

Let us consider a general second order homogeneous PDE:

$$A\frac{\partial^2 u}{\partial x^2} + B\,\frac{\partial^2 u}{\partial x \partial y} + \mathrm{C}\,\frac{\partial^2 u}{\partial y^2} + D\,\frac{\partial u}{\partial x} + E\,\frac{\partial u}{\partial y} + Fu = 0 \quad \text{where} \quad u = f(x, y)$$

or

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0 \qquad (1)$$

Equation (1) will be:

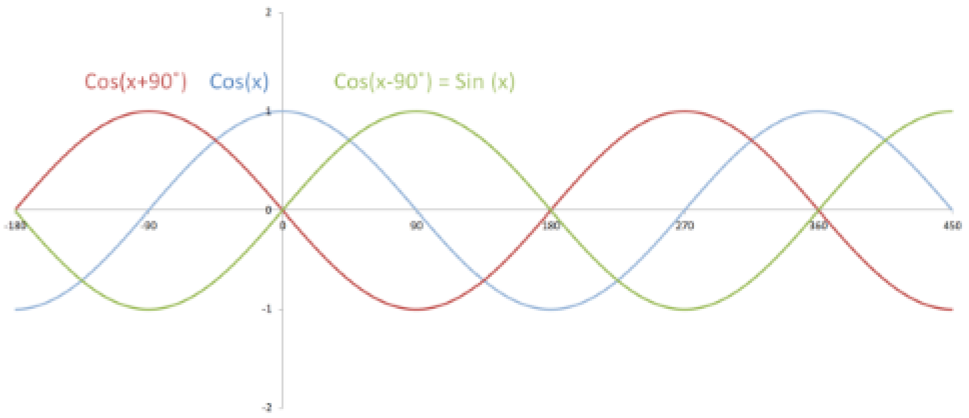
1. Hyperbolic iff $(B^2 - 4AC) > 0$

2. Parabolic iff $(B^2 - 4AC) = 0$

3. Elliptic iff $(B^2 - 4AC) < 0$

Note: Coefficients of second order partial derivatives only decide nature of PDE.

<u>Wave Equation</u>



<u>Equation:</u> $\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$

<u>Nature:</u> Hyperbolic

<u>Solution:</u> **1.** $u(x,t) = (ax+b)(ct+d)$

2. $u(x,t) = (ae^{px} + be^{-px})(ce^{Cpt} + de^{-Cpt})$

or

$u(x,t) = (a \cosh px + b \sinh px)(c \cosh Cpt + d \sinh Cpt)$

3. $u(x,t) = XT = (a \cos px + b \sin px)\,(c \cos Cpt + d \sin Cpt)$ **(Most suitable one)**

D'Alembert's Solution of Infinitely long wave (string)

Let given wave equation be: $\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$ (1)

such that $-\infty < x < \infty, t > 0$

with initial displacement $= f(x)$ and initial velocity $= g(x)$

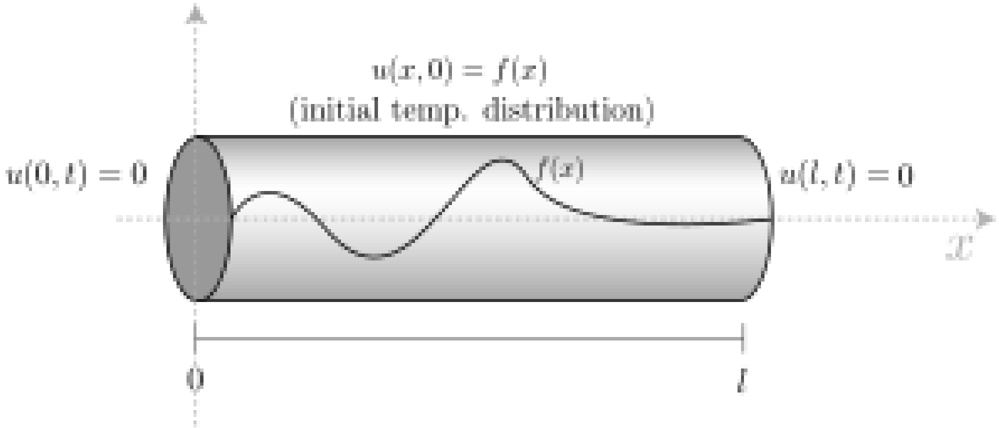
Then. D' Alembert's solution of equation (1) is given by:

$$u(x,t) = \frac{1}{2}[f(x+Ct) + f(x-Ct)] + \frac{1}{2C}\int_{x-Ct}^{x+Ct} g(s)ds$$

Heat Equation

Equation: $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$

Nature: Parabolic



Solution: **1.** $u(x,t) = Ax + B$

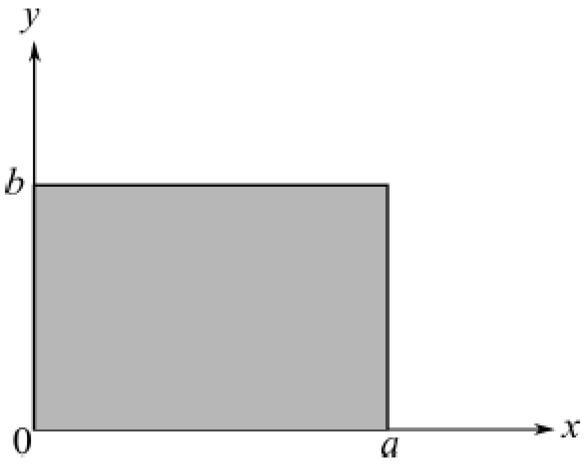
2. $u(x,t) = (Ae^{px} + Be^{-px})\, e^{C^2p^2t}$

or

$u(x,t) = (A\cosh px + B\sinh px)e^{C^2p^2t}$

3. $u(x,t) = (A\cos px + B\sin px)e^{-C^2p^2t}$ **(Most suitable one)**

Laplace Equation



Equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Nature: Elliptic

Solution: **1.** $u(x, y) = (ax + b)(cy + d)$

2. $u(x, y) = (ae^{px} + be^{-px})(c\cos py + d\sin py)$

or

$u(x, y) = (a\cosh px + b\sinh px)(c\cos py + d\sin py)$

3. $u(x, y) = (a\cos px + b\sin px)\,(ae^{py} + be^{-py})$

or

$u(x, y) = (a\cos px + b\sin px)\,(a\cosh py + b\sinh py)$

MCQ Practice Questions

In next section we will be discussing some MCQ questions from previous years question papers.

Q38. The only suitable general solution of Lapalce's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is

- (a) $u = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$ (b) $u = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-c^2 p^2 t}$
  (c) $u = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$ (d) None of these

Solution. Here option (a) is a solution of wave equation as both parts involve trigonometric cosine and sine angles.

Option (b) is a solution of Heat equation with exponential function with negative power.

Option (c) is the solution of Laplace equation which got two practically feasible solutions and it is one of those.

Hence Option **(c)** is the correct option.

Q39.Which of the following partial differential equation represents the one-dimensional heat flow equation?

(a) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (b) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (d)None of these

Solution. One-dimensional Heat equation is given by:

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$$

Hence, option **(b)** is the correct option.

Q41. Which of the following is the solution of given partial differential equation $4\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$

(a) $Ae^{\lambda(x+y)}$ (b) $A\,e^{\lambda(x-y)}$ (c) $A\,e^{\lambda(x-4y)}$ (d) $A\,e^{\lambda(4x+y)}$

Solution. We have already done this question in Lecture-19, slide-5

Hence option **(c)** is correct option.

Q42. The partial differential equation $\frac{\partial^2 u}{\partial x \partial y} = 3\frac{\partial u}{\partial x}$ is classified as

(a) Elliptic (b) Hyperbolic (c) Parabolic (d) None of these

Solution. Given equ. is: $u_{xy} - 3u_x = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

We have $A = 0, B = 1, C = 0$

Here $(B^2 - 4AC) = (1)^2 - 4(0)(0) = 1 > 0$

So, equation (1) is Hyperbolic.

Hence, option **(b)** is correct option.

Q43. If $z = f\left(\frac{x}{y}\right)$, a is any fixed constant and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial x}$. Then the partial differential equation of minimum order satisfied by z is

(a) $p = aq$ (b) $z = pq$ (c) $z = ap + qb$ (d) $px + qy = 0$

Solution. Given $z = f\left(\frac{x}{y}\right)$ (1) $[u = f(x, y)]$

Differentiating (1) partially w.r.t x

$$\frac{\partial z}{\partial x} = f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) \quad \Rightarrow p = f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) \quad \Rightarrow py = f'\left(\frac{x}{y}\right) \quad (2)$$

Differentiating (1) partially w.r.t y

$$\frac{\partial z}{\partial y} = f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) \quad \Rightarrow q = f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) \quad \Rightarrow -q\frac{y^2}{x} = f'\left(\frac{x}{y}\right) \quad (3)$$

Comparing (2) and (3), we get: $\frac{p}{2x} = \frac{q}{2y}$

$\Rightarrow py = -q\frac{y^2}{x}$ $\Rightarrow \boldsymbol{px + qy = 0}$ which is the required PDE.

Hence, option **(d)** is correct option.

Q44. Which of the following is the solution of given partial differential equation $y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$

(a) $Ae^{\lambda\left(\frac{x^2-y^2}{2}\right)}$ (b) $A\,e^{\lambda\left(\frac{x^2+y^2}{2}\right)}$ (c) $A\,e^{\lambda(x^2-4y^2)}$ (d) $A\,e^{\lambda(4x^2+y^2)}$

Solution. We have already done this question in Lecture-19, slide-7

Hence option **(a)** is correct option.

Q47. Which of the following we take as a trial solution for solving the linear partial differential equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ in the method of separation of variable

(a) $X(x)Y(y)$ (b) $X(x)T(y)$ (c) $X(x)T(t)$ (d) None of these

The given equation is:

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$$

Here u is a function of variables $x\ and\ t$.

So, trial solution is: $u = \mathrm{X}(x)\mathrm{T}(t)$

Hence option **(c)** is correct option.

Q48. Differential equation $Ar + Bs + Ct + f(x, y, z, p, q) = 0$ is hyperbolic if

a. $B^2 - 4AC < 0$ b. $B^2 - AC < 0$ c. $B^2 - 4AC > 0$ d. $B^2 - AC > 0$

Solution. Comparing the given equation with:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0 \qquad (1)$$

Equation (1) will be:

1. Hyperbolic iff $(B^2 - 4AC) > 0$

2. Parabolic iff $(B^2 - 4AC) = 0$

3. Elliptic iff $(B^2 - 4AC) < 0$

Hence option **(c)** is correct option.

Q49. Differential equation $Ar + Bs + Ct + f(x, y, z, p, q) = 0$ is parabolic if

a. $B^2 - 4AC = 0$ b. $B^2 - AC = 0$ c. $B^2 - 4AC > 0$ d. $B^2 - AC > 0$

Solution. Comparing the given equation with:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0 \qquad (1)$$

Equation (1) will be:

1. Hyperbolic iff $(B^2 - 4AC) > 0$

2. Parabolic iff $(B^2 - 4AC) = 0$

3. Elliptic iff $(B^2 - 4AC) < 0$

Hence option **(b)** is correct option.

Q50. Equation $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$ *where* $(x, y) \neq (0,0)$ is

a. Parabolic b. elliptic c. hyperbolic d. None of these

Solution. Given equ. is: $u_{xx} - x^2 u_{yy} = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

We have $A = 1, B = 0, C = x^2$

Here $(B^2 - 4AC) = (0)^2 - 4(1)(x^2) = -4x^2 < 0$

So, equation (1) is Elliptic.

Hence option **(b)** is correct option.